論文内容の要旨

論文題目：　Crystal Melting and Wall Crossing Phenomena
（結晶の溶解模型と壁越え現象）

氏名　山﨑雅人

One of the most fundamental problems in theoretical physics in the 21st century is to construct a theory of quantum gravity. General relativity and quantum mechanics, which are the cornerstones of the 20st century physics, are mutually inconsistent. We need a theory of quantum gravity which unifies the two in a consistent framework.

Over decades string theory has been the most promising candidate for quantum gravity. One of the most successful predictions of string theory, as shown by Strominger and Vafa in 1996, is that string theory correctly reproduces the Bekenstein-Hawking entropy of a class of supersymmetric, extremal black holes. In statistical mechanics, entropy is given by the logarithm of the number of states, and Strominger and Vafa showed that string theory reproduces the correct number of states. Their analysis has subsequently been generalized to many other black holes. In particular, string theory now reproduces not only the semi-classical Bekenstein-Hawking entropy of general relativity, but also the subleading contributions coming from the higher curvature corrections to the Einstein-Hilbert action. This gives a rather remarkable check of string theory as a theory of quantum gravity.

However, there are many issues that remain to be solved. For example, the entropy is typically determined only by the asymptotic growth of the microstate degeneracies, in the limit of charge charges. However, we hope that string theory gives a more complete and detailed theory of the microstates, not just their asymptotic growth. This will lead to rich and yet unknown aspects of quantum gravity.

A related question is geometry at the planck scale. One of the key ideas in general relativity is the `"geometrization of physics", where the physics notion (e.g. mass) are translated and reformulated in terms of geometry (e.g. curvature). If we follow a similar path, the central question in quantum gravity is to identify the ``quantum geometry'', geometry at the planck scale.

In this thesis, we will make small steps towards these ambitious goals. Unfortunately, solving string theory in gravity backgrounds is a notoriously difficult problem. The strategy we take is to simplify the problem --- to replace the problem of gravity by a problem of gauge theory. In string theory compactifications, this corresponds to taking the Calabi-Yau manifold to be non-compact. Of course, the notion of black hole is subtle in this limit since the gravity decouples in this limit and the Newton constant becomes zero. However, part of the important data in gravity theory still remain. For example, we can still discuss entropy of black holes since we can take a scaling limit where the mass of the black hole goes to infinity, thus the entropy is kept finite. The counting of black hole microstates is now turned into a counting problem of BPS states in supersymmetric gauge theories.

The counting problem of BPS states in string theories and supersymmetric gauge theories is an important problem, even if we forget about the motivation from black hole physics. For example, they provide primary tools to test various string dualities. BPS solitons in supersymmetric gauge theories has a rich structure, and provides a classic example of fruitful collaboration between physics and mathematics. Furthermore, as we will see in later chapters there is an intimate connection with another counting problem in string theory, the topological string theory.

In the first part of this thesis, we show that when X is a toric Calabi-Yau manifold (roughly meaning that X has an action of the three-dimensional torus), we can give explicit answers to the BPS counting problem. More precisely, each of the BPS states contributing to the BPS index (defined in section 2.1) is in one-to-one correspondence with a configuration of a molten crystal, and the BPS partition function Z_{BPS} (defined in section 2.1) is the same as the statistical partition function of a crystal melting model: $Z_{\mathrm{BPS}}=Z_{\mathrm{crystal}}$.

Chapter 3 is devoted to the explanation of this these results. Remarkably, the derivation of the above formula depends on the newly developed mathematical theory, the non-commutative Donaldson-Thomas theory. The theory gives a new invariant for Calabi-Yau manifolds, which exactly coincides with the BPS index we are interested in. This means that BPS counting problem is important not only to physicists but also the mathematicians alike.

In the next chapter (chapter 4), we discuss the implication of these results to quantum gravity. We show that the thermodynamic limit of the crystal gives a projection of the shape of the mirror of the Calabi-Yau manifold. This in particular suggests that if we start from classical smooth geometry and approaches to the planck scale, the geometry gets discretized into a set of atoms. In this sense an atom in the crystal melting model is an ``an atom of space'', a discretized version of the geometry at the planck scale. We therefore see that the two problems posed earlier are now related. Each of the microstate, which is an atom of the crystal, is the discretized version of the geometry; thus the problem of identifying black hole microstates is solved by the quantum structure of geometry!

In the second half of the thesis, we move on to the discussion of the wall crossing phenomena (see section 2.2 for introduction). Wall crossing phenomena states that the BPS degeneracy jumps as we change the value of the moduli of the Calabi-Yau manifold. Wall crossing phenomena, first discussed by Cecotti and Vafa in the context of supersymmetric N=(2,2) theories in two dimensions, has a long history of more than nearly two decades. They also play important roles in the Seiberg-Witten theory and its stringy realization. In these old days, it was observed that we can derive the jump of BPS states in simple cases, but generalization seemed to be difficult.

The recent breakthrough was triggered by the paper of Kontsevich and Soibelman, who proposed a rather general formula for the jumps of BPS degeneracies, generalizing the results of Denef and Moore. Physical interpretations of the formulas were subsequently given. In chapter 6 we discuss these formulas in detail.

The wall crossing formulas can be applied to our setup, namely compactfication on the toric Calabi-Yau manifold. In particular, the example of the resolved conifold is analyzed in Jafferi-Moore and independently in Nagao-Nakajima. There it was shown that the non-commutative Donaldson-Thomas invariants discussed in chapter 3 is related by wall crossing to the commutative (ordinary) Donaldson-Thomas invariants. In physics language, this means that the crystal melting partition function is related by wall crossings to the topological string partition function. This clarifies the connection of our crystal melting model and another crystal melting model, which describes the topological vertex.

However, this is not the end of the story. First, it was observed that the BPS partition function computed by the wall crossing formulas takes a beautiful infinite product form, and there should be an intuitive explanation of these results. From the viewpoint of non-commutative Donaldson-Thomas theory, this seems miraculous: we first compute the BPS indices separately by going through complicated mathematics, and only after summing up all of them and going through combinatorics can we see that the partition function takes such a simple form. Moreover, it was not clear why the topological string theory can be related to the BPS degeneracies. Finally, as physicists we want to have an independent way of deriving the results from purely physics arguments, without relying on the mathematical results.

This is the reason why I was motivated to give a simple derivation of the wall crossing phenomena from M-theory, which is the topic of chapter 5. By lifting type IIA brane configurations to M-theory and by using the 4d/5d correspondence, the problem of counting BPS states is mapped (under certain conditions explained in section 5) to a counting problem of free M2-brane particles in five dimensions, which span the free particle Fock space. This naturally explains the infinite product form of the BPS partition function. Also, the counting problem of M2-brane particles is a generalization of the Gopakumar-Vafa argument, which explains the appearance of the topological string partition function. More precisely, we prove a formula

$$Z_{\mathrm{BPS}}= Z_{\mathrm{top}}{}^{2}\ |_{\mathrm{chamber}}.$$

The argument here is consistent with the derivations from the wall crossing formula. The bonus is that we have new mathematical predictions for non-toric examples, which can be tested by future mathematical developments.

There is a generalization of the above-mentioned story to the case of open BPS invariants. Closed BPS invariants discussed up to this point are defined by counting D2-branes wrapping on 2-cycles of the Calabi-Yau manifold. Open BPS invariants are defined by counting D2-branes wrapping disks ending on another D-branes (D4-branes). Open BPS invariants are natural generalizations of closed invariants. Moreover, they give a useful computational tool to study closed invariants for complicated geometries, as the topological vertex formaslim shows.

In chapter 7 we first give a definition of the "non-commutative topological vertex", which gives a basic building block for computing open BPS invariants. The definition uses the crystal melting model, and we perform several consistency checks of the proposal. We also discuss the wall crossing phenomena for the open BPS invariants both with respect to the open and closed string moduli, by again using the viewpoint of M-theory.

BPS state counting and wall crossing phenomena are still very active areas of research, and has been attracting more and more attention from researchers since I started research. In the final chapter 8, we close this thesis by pointing out some interesting problems which suggest directions for future research. We also collect slightly technical results in the appendices.