論文の内容の要旨

論文題目　Formulation of Uncertainty Relations between Error and Disturbance in Quantum Measurement by using Quantum Estimation Theory
(量子推定理論を用いた測定誤差と擾乱の分析とそれらが満たす不確定性関係)

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We have studied the estimation process of quantum states from the measurement outcomes, and operationally defined the error and disturbance in quantum measurement by the accuracy of the estimation. We have shown several uncertainty relations between the measurement errors of two observables, and the uncertainty relations between the error and disturbance.

In 1927, W. Heisenberg discussed a thought experiment about the position measurement of a particle by using a γ-ray microscope. He argued that more accurately the position is measured, the more strongly the momentum of the particle is disturbed by the backaction of the measurement by using semi-classical analysis, and derived a trade-off relation between the error $\varepsilon(\hat{x})$ in the measured position $\hat{x}$ and the disturbance $\eta(\hat{p}_x)$ in the momentum $\hat{p}_x$ caused by the measurement process:

$$\varepsilon(\hat{x})\eta(\hat{p}_x) \gtrsim \hbar. \tag{1}$$

This inequality epitomizes the complementarity in quantum measurements: we cannot perform a measurement of an observable without causing disturbance in its canonically conjugate observable. However, this analysis was not fully quantum mechanical.

In the early days of quantum mechanics, the Kennard-Robertson inequality

$$\sigma(\hat{A})\sigma(\hat{B}) \geq \frac{1}{2}\left|\langle[\hat{A},\hat{B}]\rangle\right| \tag{2}$$

was erroneously interpreted as a mathematical formulation of the trade-off relation between the error and disturbance in quantum measurement, where $\langle\hat{A}\rangle := \mathrm{Tr}[\hat{\rho}\hat{A}]$ is the expectation value of $\hat{A}$ over the quantum state $\hat{\rho}$, the square bracket denotes the commutator $[\hat{A},\hat{B}] := \hat{A}\hat{B}-\hat{B}\hat{A}$, and $\sigma(\hat{A})^2 := \langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2$. The Kennard-Robertson inequality actually implies the indeterminacy of the quantum state: non-commuting observables cannot have definite values simultaneously.

However, since $\sigma(\hat{A})$ does not depend on the measurement process, the Kennard-Robertson inequality reflects the inherent nature of a quantum state alone, and does not concern any trade-off relation between the error and disturbance in the measurement process.

In 1988, E. Arthurs and M. S. Goodman considered a simultaneous measurement of two non-commuting observables $\hat{A}$ and $\hat{B}$ in a fully quantum mechanical treatment. Because $\hat{A}$ and $\hat{B}$ do not commute with each other, it is necessary to extend the Hilbert space to make both of them simultaneously measurable. This can be done by letting the system interact with another system, called the apparatus. By considering the interaction between the system and apparatus, they considered an indirect measurement. In order to make the outcomes of the indirect measurement meaningful for $\hat{A}$ and $\hat{B}$, they assumed the unbiasedness of the measurement outcomes. That is, the expectation values of the outcomes respectively equal to $\langle\hat{A}\rangle$ and $\langle\hat{B}\rangle$ for an arbitrary quantum state. The unbiasedness of the measurement implies that $\langle\hat{A}\rangle$ can be estimated directly from the distribution of the measurement outcomes. They derived that the variances of the measurement outcomes satisfy

$$\sigma'(\hat{A})\sigma'(\hat{B}) \geq \left|\langle[\hat{A},\hat{B}]\rangle\right|. \tag{3}$$

Comparing this result with the Kennard-Robertson inequality, we find that the lower bound is doubled. In the model of Arthurs nad Goodman, the sources of the fluctuations of the measurement outcomes are both the system's inherent fluctuations and the error in the measurement process. Each source of the fluctuations has the lower bound of $\frac{1}{2}\left|\langle[\hat{A},\hat{B}]\rangle\right|$, and the bound in (3) is doubled. Because the measurement discussed by Arthurs and Goodman is restricted to the unbiased measurement, a natural question arises as to what happens for the biased measurement case.

M. Ozawa generalized the Arthurs-Goodman inequality by removing the unbiasedness condition, and derived the following inequality:

$$\varepsilon(\hat{A})\varepsilon(\hat{B}) + \varepsilon(\hat{A})\sigma(\hat{B}) + \sigma(\hat{A})\varepsilon(\hat{B}) \geq \frac{1}{2}\left|\langle[\hat{A},\hat{B}]\rangle\right|, \tag{4}$$

Because the error $\varepsilon(\hat{A})$ is always finite, if the error $\varepsilon(\hat{A})$ vanishes, the product of the measurement errors also vanishes. Thus, the Heisenberg-type trade-off relation can be violated:

$$\varepsilon(\hat{A})\varepsilon(\hat{B}) = 0 \leq \frac{1}{2}\left|\langle[\hat{A},\hat{B}]\rangle\right|. \tag{5}$$

However, his definition of the measurement error does not correspond to the accuracy of the estimation any more due to the removal of the unbiasedness condition. For example, if the outcome of a measurement is fixed for an arbitrary state $\hat{\rho}$, the error $\varepsilon(\hat{A})$ can vanish even if we cannot estimate $\langle\hat{A}\rangle$. Such a result originates from ignoring the estimation process which must inevitably be accompanied in the unbiased measurement. Ozawa also defined the disturbance $\eta(\hat{B})$ caused by the backaction of the measurement, and derived the following inequality:

$$\varepsilon(\hat{A})\eta(\hat{B}) + \varepsilon(\hat{A})\sigma(\hat{B}) + \sigma(\hat{A})\eta(\hat{B}) \geq \frac{1}{2}\left|\langle[\hat{A},\hat{B}]\rangle\right|. \tag{6}$$

However, his definition of the disturbance does not involve the estimation process either. We consider that to analyze the error and disturbance in quantum measurement, it is crucial to clarify the role of the estimation which must be made on the measurement outcomes.

Estimation theory provide us a description of how accurately we can estimate values and how much information we can obtain from realizations of the probabilistic phenomena. In quantum theory, measurements on the quantum system are necessary to obtain some pieces of information about the quantum system, and the measurement outcomes are obtained according to the probability distribution. Thus, it is necessary to involve the estimation theory for clarifying the uncertainty relations about the error and disturbance in quantum measurement. In estimation theory, one of the most important quantities is the Fisher information, which gives the upper bound on the accuracy of the estimated value. The probability distribution of the outcomes is determined by the measured quantum state and the choise of the measurement operators. The probability distribution and the Fisher information vary with varying the measurement. The quantum Fisher information gives the upper bound of the Fisher information obtained by the measurement.

We develop a general theory of the error and disturbance in quantum measurement from the viewpoint of quantum estimation theory. We show that the unbiasedness is not necessary for the measurements, but for the estimation from the measurement outcomes. We analyze that the estimated values from measurement outcomes consist of three types of errors: inherent fluctuation of an observable, error in the measurement, and the estimation error caused by the unoptimality of the estimator. We extract the measurement error from the variance of the estimator in terms of the Fisher information, which gives the upper bound of the accuracy of the estimator. Our definition of the measurement error reduces to Arthurs-Goodman's definition for the case of the unbiased measurement. We also analyze that disturbance caused by the backaction of the measurement is quantified in terms of the loss of the Fisher information contents during the measurement process.

By using our definition of the error and disturbance, we prove that the following uncertainty relation between the errors of two observables

$$\varepsilon(\hat{A})\varepsilon(\hat{B}) \geq \frac{1}{4}\left|\langle[\hat{A},\hat{B}]\rangle\right|^2, \tag{7}$$

and the uncertainty relation between the error and disturbance

$$\varepsilon(\hat{A})\eta(\hat{B}) \geq \frac{1}{4}\left|\langle[\hat{A},\hat{B}]\rangle\right|^2. \tag{8}$$

However, the lower bounds, in general, cannot be attained in these inequalities. By introducing quantities $\sigma_Q(\hat{A})$ and $\mathcal{C}_{\mathrm{QS}}(\hat{A},\hat{B})$, which respectively characterize the quantum fluctuations of $\hat{A}$ and the quantum correlation between $\hat{A}$ and $\hat{B}$, we derive new inequalities providing the attainable bounds:

$$\varepsilon(\hat{A})\varepsilon(\hat{B}) \geq \sigma_Q(\hat{A})^2\sigma_Q(\hat{B})^2 - \mathcal{C}_{\mathrm{QS}}(\hat{A},\hat{B})^2, \tag{9}$$

$$\varepsilon(\hat{A})\eta(\hat{B}) \geq \sigma_Q(\hat{A})^2\sigma_Q(\hat{B})^2 - \mathcal{C}_{\mathrm{QS}}(\hat{A},\hat{B})^2. \tag{10}$$

If we focus on the observable $\hat{A}$ and $\hat{B}$, they are inherently fluctuated in general. Such fluctuations can be separated to the two parts: classical fluctuations and quantum fluctuations. The value $\sigma_Q(\hat{A})$ characterizes the fluctuation which originate from the quantum mechanics. The quantum correlation $\mathcal{C}_{\mathrm{QS}}(\hat{A},\hat{B})$ also characterizes the correlation of the observable which originate form the quantum mechanics. We have proved that these bounds can be attained for the 2-dimensional Hilbert space, and for a class of measurements for higher dimensions. We have shown numerical evidences for the validity of the bounds for an arbitrary measurement for the higher-dimensional Hilbert spaces.